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ORIGINAL PAPER

On representation of monotone preference orders in a sequence space

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Abstract In this paper we investigate the relation between *scalar continuity* and representability of *monotone* preference orders in a sequence space. Scalar continuity is shown to be sufficient for representability of a monotone preference order and easy to verify in concrete examples. Generalizing this result, we show that a condition, which restricts the extent of scalar discontinuity of a monotone preference order, ensures representability. We relate this condition to the well-known *order dense property*, which is both necessary and sufficient for representability.

1 Introduction

In this paper, we consider the problem of representability of *monotone* preference orders on a *sequence space*.[1] Monotone preferences are especially compelling in the theory of social evaluation of intertemporal utility streams since if no one is worse off, then the society as a whole should not be worse off (see Diamond 1965, p. 172). However, monotone preferences (expressing that "more is better") have also been used in the theory of individual preferences on commodity bundles, at least since the study of Wold (1943).[2]

[1] We consider both finite and infinite sequence spaces as the theory developed here holds regardless of the dimension of the space.

[2] For a comprehensive discussion of Wold's result on the existence of a continuous utility function representing a preference order, see Beardon and Mehta (1994).

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The general characterization of representability of a preference ordering is the *order dense* property,[3] and this, of course, applies to our setting.[4] However, as is well known, the order dense property can be difficult to apply to concrete examples to decide on the representability (or non-representability) of a preference ordering. Thus, our objective is to present a sufficient condition for representability which provides a *partial characterization* but which is relatively easy to check in concrete examples.

We focus on a novel concept of *scalar continuity* of preferences, which may be described as follows. Given any utility stream x, consider the set of *constant* utility streams which are at least as good as x, and the set of *constant* utility streams which are at most as good as x. These sets can be identified with sets of scalars, since constant utility streams are scalar multiples of the constant utility stream with constant utility equal to one. The preference order is scalar continuous if these sets of scalars are closed subsets of the real line. Thus scalar continuity requires continuity of preferences on the *diagonal* of the space of utility streams, and is therefore easy to verify. One of our main results (Proposition 2) is that a monotone, scalar continuous preference order can always be represented by a real valued function.

Here is a brief outline of the contents of the paper. Section 2 introduces the notation and definitions. In Sect. 3, we associate with any monotone preference order a *pseudo utility function* μ which provides a *weak representation* of the order.[5] No continuity condition is imposed to obtain this pseudo utility function.

In Sect. 4, we present our main representation results. In Proposition 2, we show that if the order satisfies a *scalar continuity* condition, then it is representable by the function μ.[6] Generalizing this result, we establish in Theorem 1 that when the set of equivalence classes (indifference curves) which have points of scalar discontinuity, is countable, then there exists a representation for the order. We indicate how *countable scalar discontinuity* condition can be used to verify the order dense property. However, this condition is not equivalent to the order dense property; an example is given to show that countable scalar discontinuity is *not* necessary for representability of a monotone preference order.

In Sect. 5, we consider a number of representation results which have been developed in the literature. We show that these results can be derived easily by applying Proposition 2 and Theorem 1.

Finally, in Sect. 6 we conclude and in an Appendix, we provide two examples to illustrate the relationship between our concept of scalar continuity, and the concepts of sup-norm continuity and restricted continuity used in the literature.

[3] Order dense property requires the existence of a countable subset which is dense in its given superset with respect to the underlying order topology. For a formal definition of this notion, see Remark 1, p. 8.

[4] For expositions of this characterization result, see Fishburn (1970), Kreps (1988) and Bridges and Mehta (1995).

[5] This terminology follows Peleg (1970).

[6] This terminology follows Weibull (1985), who uses a similar concept of continuity.

2 Notation and definitions

Let $\mathbb{N}$ denote, as usual, the set of natural numbers $\{1, 2, 3, \ldots\}$, and let $\mathbb{R}$ denote the set of real numbers. Let Y denote the closed interval $[0, 1]$, and let X denote the set Y^M where $M \in \mathbb{N} \cup \{\infty\}$.[7] Thus, $x \in X$ if and only if $x_n \in [0, 1]$ for all $n \in \mathbb{N}$ such that $n \leq M$ for some $M \in \mathbb{N} \cup \{\infty\}$. One can interpret x_n as the utility level of generation n, and x as an infinite stream of these utility levels or x_n as the consumption level of good n, and x as a finite bundle of these consumption levels.

The constant sequence of zeros in X will be denoted by 0, and the constant sequence of ones in X will be denoted by e. We denote the set of all constant sequences in X, $\{\lambda e \in X : \lambda \in [0, 1]\}$, by C and we call it the *diagonal* in X.

For $y, z \in \mathbb{R}^M$ with $M \in \mathbb{N} \cup \{\infty\}$, we write $y \geq z$ if $y_i \geq z_i$ for all $i \in \mathbb{N}$ such that $i \leq M$; $y > z$ if $y \geq z$, and $y \neq z$, and $y \gg z$ if $y_i > z_i$ for all $i \in \mathbb{N}$ such that $i \leq M$.

A *preference ordering* is a binary relation, $\succsim$ on X, which is complete and transitive. We associate with $\succsim$ its asymmetric and symmetric components by $\succ$ and $\sim$ respectively.

A preference ordering $\succsim$ on X is called *monotone* (M) if the following condition holds:

(M) If $x, y \in X$ and $x \geq y$, then $x \succsim y$.

We say that a preference ordering $\succsim$ on X is *strongly monotone* (SM) if it satisfies the following efficiency condition:

(SM) If $x, y \in X$, with $x > y$, then $x \succ y$.

When x and y are utility streams, then (SM) is the standard Pareto principle. Clearly the former efficiency condition is implied by the latter one. Strong monotonicity also implies the following condition called *weak Pareto*:

(WP) If $x, y \in X$ and $x \gg y$, then $x \succ y$.

A preference ordering $\succsim$ on X is *representable* if there is a function, $u : X \to \mathbb{R}$, such that for all $x, y \in X$,

$$x \succsim y \quad \text{if and only if } u(x) \geq u(y) \tag{R}$$

Given a preference ordering $\succsim$ on X, for each $x \in X$ the *Lower and Upper Contour Sets* are defined as $LC(x) = \{y \in X : x \succsim y\}$ and $UC(x) = \{y \in X : y \succsim x\}$ respectively.

Given a topology $\mathcal{T}$ for X, we say that $\succsim$ is $\mathcal{T}$-*continuous* on X if for each $x \in X$, the lower and upper contour sets ($LC(x)$ and $UC(x)$) of x are *closed* subsets of X in the topology $\mathcal{T}$.

7 We use this notation to accommodate both the finite dimensional and infinite dimensional sequence spaces.

3 Weak representation of a monotone preference order

We will associate with a monotone preference order $\succsim$ on X a *pseudo utility* function $\mu : X \to \mathbb{R}$ which provides a *weak representation* of it; that is,

$$\text{If } x, y \in X \text{ and } x \succsim y, \text{ then } \mu(x) \geq \mu(y) \tag{WR}$$

Condition WR implies that if $x, y \in X$ and $x \sim y$, then $\mu(x) = \mu(y)$. Also, if $x, y \in X$ and $\mu(x) > \mu(y)$, then $x \succ y$. However, it allows for the possibility that $x, y \in X$ satisfy $x \succ y$, but $\mu(x) = \mu(y)$. It is in this respect that the representation is weak.

For each $x \in X$ define the following subsets of the $[0, 1]$ interval:

$$A(x) = \{\lambda \in [0, 1] : \lambda e \succsim x\}; \;\; B(x) = \{\lambda \in [0, 1] : x \succsim \lambda e\} \tag{1}$$

Note that while upper and lower contour sets are subsets of X, the sets $A(x)$ and $B(x)$ are subsets of the real line.

Proposition 1 *Let $\succsim$ be a monotone preference ordering on X. Then, $\succsim$ has a weak representation.*

Proof We obtain a weak representation as follows. For each $x \in X$, define $A(x)$ as in (1) and let $\alpha(x)$ be the infimum of the set $A(x)$. That is:

$$\alpha(x) = \inf_{\lambda \in A(x)} \lambda \quad \text{for each } x \in X \tag{2}$$

Note that since $\succsim$ is monotone, $A(x)$ is non-empty and so $\alpha(x)$ is well-defined. Clearly, $\alpha(x) \in [0, 1]$ for each $x \in X$. We claim that $\alpha(x)$ satisfies condition WR. Let $x, y \in X$ satisfy $x \succsim y$. By (1), we have $A(x) \subset A(y)$ and thus $\alpha(x) \geq \alpha(y)$ by the monotonicity of an infimum. □

Note that α is not the only possible weak representation of $\succsim$ on X. For instance, one can define $B(x)$ as in (1) and let $\beta(x)$ be the supremum of the set $B(x)$ for each $x \in X$. That is:

$$\beta(x) = \sup_{\lambda \in B(x)} \lambda \quad \text{for each } x \in X \tag{3}$$

Since $\succsim$ is monotone, $B(x)$ is non-empty and thus $\beta(x)$ is well-defined. Clearly, $\beta(x) \in [0, 1]$ for each $x \in X$. Moreover, if $x, y \in X$ with $x \succsim y$, then $B(x) \supset B(y)$ by (1) and so $\beta(x) \geq \beta(y)$ by the monotonicity of a supremum. Therefore, β satisfies condition WR and is a weak representation of $\succsim$ on X.

In general, these two functions, α and β, need not be equal. However, $\alpha(x)$ can be at most $\beta(x)$ for each $x \in X$. For if $\alpha(x) > \beta(x)$ for some $x \in X$, then we can pick some $\theta \in (\beta(x), \alpha(x))$. Since $\succsim$ is complete and thus $A(x) \cup B(x) = [0, 1]$ by (1), $\theta \in (0, 1)$ must belong to $A(x)$ or $B(x)$. However, if $\theta \in A(x)$, we must have

$\theta \geq \alpha(x)$ by (2), a contradiction. And, if $\theta \in B(x)$, we must have $\theta \leq \beta(x)$ by definition of $\beta(x)$, a contradiction. Thus we have:

$$\alpha(x) \leq \beta(x) \quad \text{for all } x \in X \tag{4}$$

In addition, by using α and β one can define many similar functions as well to serve as weak representations for the preference order $\succsim$. To see this, let $k \in (0, 1)$ and define the function $\mu_k : X \to [0, 1]$ as follows:

$$\mu_k(x) = k\alpha(x) + (1-k)\beta(x) \quad \text{for all } x \in X \tag{5}$$

Then if we let $x, y \in X$ with $x \succsim y$, we get $\mu_k(x) = k\alpha(x) + (1-k)\beta(x) \geq k\alpha(y) + (1-k)\beta(y) = \mu_k(y)$ exhibiting WR.

We note here that if the preference order is monotone and satisfies Weak Pareto, then it can be shown that $\alpha(x) = \beta(x) = \mu_k(x)$ for all k in $(0, 1)$. However, monotone preference orders satisfying Weak Pareto need not be representable. The well-known example of the lexicographic preference order (see Debreu 1954) satisfies Strong Monotonicity, and therefore is a monotone preference order satisfying Weak Pareto, but is not representable.[8]

4 Representation of a monotone preference order: a sufficient condition

In this section, we use the weak representation result of Sect. 3 to provide a representation for monotone preference orders. For this purpose, we use a weak notion of continuity of preferences, called scalar continuity, to present our first representation result (Proposition 2). We then generalize this result to cover cases in which preference orders might exhibit a limited extent of scalar discontinuity (Theorem 1).

The following lemma is useful in obtaining our representation results.

Lemma 1 *Let $\succsim$ be a monotone preference ordering on X. Suppose x in X is a point such that the sets $A(x)$ and $B(x)$ defined in* (1) *have a non-empty intersection. Then $\mu_k(x)e \sim x$ where μ_k is defined in* (5).

Proof Let x be in X be such that $A(x) \cap B(x)$ is non-empty. We have $\alpha(x) \leq \beta(x)$ by (4) and so we have two cases to consider; (i) $\alpha(x) < \beta(x)$, (ii) $\alpha(x) = \beta(x)$. In case (i), using (5) we get $\alpha(x) < \mu_k(x) < \beta(x)$. Then by (2) and (3), we have $\mu_k(x) \in A(x) \cap B(x)$ and so by (1), $x \sim \mu_k(x)e$. In case (ii), we have $\alpha(x) = \mu_k(x) = \beta(x)$ by (5). Thus given that $A(x) \cap B(x)$ is non-empty and using (2) and (3), for any $\lambda \in A(x) \cap B(x)$, we get $\alpha(x) \leq \lambda \leq \beta(x)$, and so $\lambda = \mu_k(x)$. This shows that $\mu_k(x) \in A(x) \cap B(x)$ and therefore by (1), we infer that $x \sim \mu_k(x)e$. □

The following *weak* notion of continuity will be used in the next representation result.

[8] For a comprehensive study of complete preference orders which are not representable by a real valued function, see Beardon et al. (2002). As the characterization result in that paper indicates, these turn out to be of four types; an open question is which types can occur for complete preference orders which are monotone.

Definition 1 We say that $\succsim$ is *scalar continuous* on X if for each $x \in X$, the sets $A(x)$ and $B(x)$ defined in (1) are closed in the standard topology on $\mathbb{R}$.

Sup-norm continuity (see Example 3 in Sect. 5) implies restricted continuity (see Example 4 in Sect. 5), which in turn implies scalar continuity. Further, as is shown in Example 1 in the Appendix, there are monotone preference orders which satisfy scalar continuity, but violate restricted continuity and (therefore) sup-norm continuity. And, as shown in Example 2 in the Appendix, there are monotone preference orders which satisfy restricted continuity but violate sup-norm continuity.

We can now state the following representation result for monotone preference orders.

Proposition 2 *If a monotone preference ordering* $\succsim$ *is scalar continuous on* X, *then* $\succsim$ *is representable, and* μ_k, *defined in* (5), *represents it.*

Proof We know by (5) that μ_k is a weak representation of $\succsim$. Thus it remains to show that when $x, y \in X$ and $x \succ y$, we must have $\mu_k(x) > \mu_k(y)$.

Since $\succsim$ is scalar continuous, $A(x)$ and $B(x)$ are closed subsets in [0, 1] for each $x \in X$. Moreover, since $\succsim$ in X is complete, then the union $A(x) \cup B(x)$ exhausts the interval [0, 1] which is a connected set. Thus, $A(x) \cap B(x)$ must be non-empty for all $x \in X$. Thus, by Lemma 1 we have $\mu_k(x)e \sim x \succ y \sim \mu_k(y)e$ so that $\mu_k(x)e \succ \mu_k(y)e$ by transitivity. This implies $\mu_k(x) \neq \mu_k(y)$, and since by (WR), $\mu_k(x) \geq \mu_k(y)$, we must have $\mu_k(x) > \mu_k(y)$. □

4.1 A refinement of the representation result

For any preference order $\succsim$ on X, let:

$$D = \{x \in X : A(x) \cap B(x) = \emptyset\} \tag{6}$$

When $x \in D$, we refer to it as a *point of scalar discontinuity* of the preference order $\succsim$ on X. When D is empty, we say that the order $\succsim$ has no points of scalar discontinuity in X.

The ability to represent $\succsim$ depends crucially on (loosely speaking) "how many" points of scalar discontinuity there are. To make this notion precise, we make the preliminary remark that if $x \in X$ is a point of scalar discontinuity, and $y \sim x$, then y is also a point of scalar discontinuity. To see this, suppose $x \in D$. Since $y \sim x$, we have $A(x) = A(y)$ and $B(x) = B(y)$, and so $A(y) \cap B(y) = \emptyset$ implying that $y \in D$.

In view of this remark, proceed to form the following partition of D. For each $x \in D$, let $E(x) = \{z \in X : z \sim x\}$; clearly $E(x)$ is non-empty since $x \in E(x)$. For all $x, y \in X$, we have either $E(x)$ disjoint from $E(y)$, or $E(x) = E(y)$; further:

$$\bigcup_{x \in D} E(x) = D$$

Let $\Im$ be the collection $\{E(x) \text{ for some } x \in D\}$. Then, $\Im$ is a partition of D. In order to see "how much" scalar discontinuity the preference order exhibits, it is enough to look at "how many" equivalence classes there are in $\Im$. We can now introduce the following condition:

Countable scalar discontinuity condition:
The collection $\Im$ has at most a countable number of equivalence classes.

We now show that this countable scalar discontinuity condition is *sufficient* for the representability of a monotone preference order.

Theorem 1 *If a monotone preference ordering $\succsim$ on X satisfies the countable scalar discontinuity condition, then $\succsim$ is representable.*

Proof Let $\{F_1, F_2, \ldots\}$ be an enumeration of the set $\Im$. For each $F_n \in \Im$, let $r(F_n) = (1/2^n)$. Let $\mathbb{F}$ be the collection of all subsets of $\Im$ and define a function $\pi : \mathbb{F} \to \mathbb{R}$ as follows:

$$\pi(\Im') = \begin{cases} \sum_{F_n \in \Im'} r(F_n) & \text{if } \Im' \text{ is non-empty} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Note that since the sequence $\{r(F_n)\}$ is summable, π is well-defined, and indeed for any $\Im' \in \mathbb{F}$, $\pi(\Im') \in [0, 1]$.

For each $x \in X$, let $W(x) = \{y \in X : x \succ y\}$ and $P(x) = \{y \in X : y \succ x\}$. Note that for any $x \in X$, the sets $W(x)$ and $P(x)$ together separate $\Im$ into its subsets as for any $F_n \in \Im$, either $F_n \subset W(x)$ or $F_n \subset P(x)$ or $F_n \cap [W(x) \cup P(x)] = \emptyset$. For any $x \in X$, let $\Im_*(x) = \{F_n \in \Im : F_n \subset W(x)\}$ and $\Im^*(x) = \{F_n \in \Im : F_n \subset P(x)\}$, and define the function:

$$\rho(x) = \pi(\Im_*(x)) - \pi(\Im^*(x)) \tag{8}$$

Since π is bounded on $\mathbb{F}$, ρ is well-defined. Now consider the function $u : X \to \mathbb{R}$ defined as:

$$u(x) = \mu_k(x) + \rho(x) \tag{9}$$

where μ_k is defined in (5). We claim that u represents the ordering $\succsim$ on X.[9]

Let $x, y \in X$ such that $x \succsim y$. Then by the definition of the sets $\Im_*$ and $\Im^*$, we have $\Im_*(y) \subset \Im_*(x)$ and $\Im^*(x) \subset \Im^*(y)$. Thus by (7) and (8), we get $\rho(x) \geq \rho(y)$ and so by (5) and (9), $u(x) \geq u(y)$.

Now let $x, y \in X$ such that $x \succ y$. By (5) we have $\mu_k(x) \geq \mu_k(y)$. If $\mu_k(x) > \mu_k(y)$, then we have $u(x) > u(y)$ by (9) and the fact that $\rho(x) \geq \rho(y)$. If, however, $\mu_k(x) = \mu_k(y)$, then $x \in D$ or $y \in D$ must hold. Otherwise, by Lemma 1 we have $x \sim y$, a contradiction. In both cases, $x \in D$ or $y \in D$, one of the two set inclusions, $\Im_*(y) \subset \Im_*(x)$ and $\Im^*(x) \subset \Im^*(y)$, must be strict and so we must have $\rho(x) > \rho(y)$. Thus, by (9) $u(x) > u(y)$ which establishes the claim. □

9 Our representation shows a link between measure and utility theory as the function π, defined in (7), is a simple measurable function. On a general approach using measure theory in constructing a representation, see Voorneveld and Weibull (2009).

Remark 1 Using the countable scalar discontinuity condition, one can directly check the *order dense property* for any monotone preference order; that is, one can find a countable subset Z which is *order dense* in X.[10] To see this, let $Q = \{qe \in X$: for some rational number $q\}$. Then $Q \subset X$ is countable. By countable scalar discontinuity, $\Im$ is countable and so by using the *Axiom of Denumerable Choice*,[11] one can pick an element from each $E \in \Im$; call this $g(E)$. Then the set $F = \{g(E) \in X : E \in \Im\}$ is also countable and so $Z = Q \cup F$ is a countable subset of X. We now verify that Z is order dense in X.

Let $x, y \in X$, with $x \succ y$. There are two cases to consider; (i) either x or $y \in D$ or (ii) neither x nor $y \in D$. In case (i), without loss of generality, let $x \in D$. Then $z = g(E(x)) \in F$ and $x \sim z$. Thus we have $x \succsim z \succsim y$. In case (ii), we have $\mu_k(x) > \mu_k(y)$ by Lemma 1 and so there exists $z = qe \in Q$ such that $\mu_k(x) > q > \mu_k(y)$. By monotonicity of the order, we then have $\mu_k(x)e \succsim qe \succsim \mu_k(y)e$ and so $x \succsim z \succsim y$. This shows that Z is order dense in X.[12]

Remark 2 A topic that has been discussed extensively in the social choice literature is the possible incompatibility of an efficiency concept like Strong Pareto with an equity concept like Anonymity when $X = Y^\infty$. Basu and Mitra (2003) showed that any preference order satisfying Strong Pareto and Anonymity cannot be represented by a real valued function. Further, although Svensson (1980) showed that preference orders satisfying Strong Pareto and Anonymity exist, the results of Zame (2007) and Lauwers (2010) imply that such preferences cannot be *constructed* and require the use of the Axiom of Choice or similar contrivance for demonstrating their existence.

Our Theorem 1 implies that any preference order satisfying Strong Pareto and Anonymity must have an uncountable number of equivalence classes, which have points of scalar discontinuity. Recall that if $x \in X$ is a point of scalar discontinuity, then there is no $\lambda \in [0, 1]$ such that λe is indifferent to x. [For, if there were such a λ, then this λ would belong to both $A(x)$ and $B(x)$, contradicting the fact that x is a point of scalar discontinuity.] Thus, there is an uncountable number of indifference curves, generated by such a preference order, which are disjoint from the diagonal of X. This provides further insight about the nature of efficient and equitable preference orders on infinite utility streams.

An example

We now present an example in $X = Y^2$ to show that the countable scalar discontinuity condition is *not a necessary condition* for representability of a monotone preference order $\succsim$. This also shows that for monotone preference orders, countable scalar discontinuity is not equivalent to the order dense property. Thus, Theorem 1 is only a partial characterization of the representability of a monotone preference order.

10 Z is order dense in X (in the sense of Debreu) if $x, y \in X$ and $x \succ y$ imply that there is some $z \in Z$, such that $x \succsim z \succsim y$ (See Bridges and Mehta 1995, pp. 11, 12).

11 This axiom of set theory is a weak form of the Axiom of Choice. For an exposition of the Axiom of Choice, see Munkres (1975, p. 59).

12 Thus the result of Theorem 1 also follows by appealing to the order dense characterization result on representability if one grants the Axiom of Denumerable Choice.

Let us define $u(x_1, x_2)$ for all $(x_1, x_2) \in X = Y^2$ as follows:

$$u(x_1, x_2) = \begin{cases} x_1 & \text{for } x_1 \in [0, (1/2)) \\ (1/2) + x_2 & \text{for } x_1 = (1/2) \\ 1 + x_1 & \text{for } x_1 \in ((1/2), 1] \end{cases} \tag{10}$$

Then, define $\succsim$ on X as follows. For $x, y \in X$, $x \succsim y$ if and only if $u(x) \geq u(y)$. Then $\succsim$ is clearly a preference order, and it is monotone.

Let $U = \{x \in X : x_1 = (1/2) \text{ and } x_2 \in ((1/2), 1]\}$. Note that whenever $x \in U$, we have $A(x) = ((1/2), 1]$ and $B(x) = [0, (1/2)]$. Thus $D \supset U$. Moreover, for any $x, x' \in U$, we have $x \sim x'$ if and only if $x = x'$. Thus, $\Im \supset \{\{x\} : x \in U\}$ and so $\Im$ is uncountable. But, (10) is clearly a representation of $\succsim$ on X.

5 Applications of the representation result

We now consider four different representation results which have been derived in the literature. Our aim, here, is to show that each example below provides sufficient conditions to derive scalar continuity and/or countable scalar discontinuity conditions, and thereby to demonstrate that these well-known representation results follow from our results established in Sect. 4.

Example 1: Wold's representation

The first theory on the existence of a continuous representation for a preference order was given in a fundamental paper of Wold (1943). Wold considers strongly monotone preference order, and establishes a representation by showing that every indifference class meets the diagonal given the condition below. We show how this result can be obtained by applying Proposition 2.

Let $X = Y^n$ for some $n \in \mathbb{N}$ and let $\succsim$ be a strongly monotone preference order on X satisfying the following continuity condition:

Wold: For any $x, y, z \in X$ such that $x \succ y \succ z$, there exist some $a, b \in (0, 1)$ such that $ax + (1 - a)z \succ y \succ bx + (1 - b)z$.

Using strongly monotone preferences, $x \sim 0$ if and only if $x = 0$, and $x \sim e$ if and only if $x = e$. Further $A(0) = [0, 1]$, $B(0) = \{0\}$, and $A(e) = \{1\}$, $B(e) = [0, 1]$. Thus, for $x \sim 0$, $A(x)$ and $B(x)$ are closed sets; and for $x \sim e$, $A(x)$ and $B(x)$ are closed sets. Consider then any $x \in X$, such that $e \succ x \succ 0$. We show that $A(x)$ is closed as follows. Let $\{\lambda^s\}_{s=1}^{\infty}$ be a convergent sequence of elements in $A(x)$, converging to λ^0. We have to show that $\lambda^0 \in A(x)$. If this is not the case, then we have $\lambda^0 \in [0, 1]$, and $e \succ x \succ \lambda^0 e$. So, $\lambda^0 < 1$, and by Wold's condition, there is $b \in (0, 1)$, such that $x \succ [b\lambda^0 + (1-b)]e$. Since $\lambda^s \to \lambda^0$ as $s \to \infty$ and $\lambda^0 < [b\lambda^0 + (1-b)]$, we can find s' large enough for which $\lambda^{s'} < [b\lambda^0 + (1 - b)]$. But, then, $x \succ \lambda^{s'} e$, a contradiction to the fact that $\lambda^s \in A(x)$ for all $s \in \mathbb{N}$. Thus, $\lambda^0 \in A(x)$, and $A(x)$ is closed. We can show that $B(x)$ is closed by following a similar line of proof. Thus, the preference order $\succsim$ satisfies scalar continuity and is representable, by using Proposition 2.

If the preference order is monotone but not strongly monotone, the sets $A(x)$ and $B(x)$ need not be closed for all $x \in X$. Consider the preference relation $\succsim$ on X for which $x \sim e$ if $x_n > 0$ for some $n \in \mathbb{N}$, and $e \succ 0$. This is easily seen to be a preference order, which is monotone. Further, Wold's condition is trivially satisfied since one cannot find three points x, y, z in X satisfying $x \succ y \succ z$. Now, if $x \in X$ with $x \neq 0$, then $A(x) = (0, 1]$, so $A(x)$ is not closed. Thus, $\succsim$ does not satisfy scalar continuity, and Proposition 2 is not applicable.

However, if the preference order is monotone, and Wold's condition is satisfied, we can still show that the preference order is representable by using Theorem 1 as follows. By using Wold's condition, one can show the following fact (see Fishburn 1970, p. 33).

Fact: If $x, y, z \in X$ such that $x > y > z$, then there exists some $a \in [0, 1]$ such that $y \sim ax + (1 - a)z$.

We claim that there is no point of scalar discontinuity of $\succsim$ and thus by Theorem 1, $\succsim$ is representable. To see this, let $y \in X$. We have $e \geq y \geq 0$. If $y = 0$ or $y = e$, then $A(y) \cap B(y)$ is non-empty. If $y \neq 0$ and $y \neq e$, then $e > y > 0$. Then by the fact above, there is some a in $[0, 1]$ such that $y \sim ae + (1 - a)0 = ae$ showing that $A(y) \cap B(y)$ is non-empty for any $y \in X$.

Example 2: weighted utilitarian representation

In the theory of social choice, one of the prominent judgment criteria on the welfare of the society is called (weighted) utilitarianism. This method seeks to maximize the society's collective welfare obtained by summing (weighted) individual utilities [see d'Aspremont and Gevers (2002) for a discussion of available characterization results]. We show below how one can achieve the existence of a representation for a preference order satisfying a set of axioms used in this literature.

Let $X = Y^n$ for some $n \in \mathbb{N}$ and let $\succsim$ be a monotone preference order on X satisfying WP and the following two conditions:

Minimal individual symmetry (MIS): For all $i, j \in \{1, 2, \ldots, n\}$, there exist $x, y \in X$ such that $x_i > y_i$, $x_j < y_j$, $x_k = y_k$ for all $k \in \{1, 2, \ldots, n\} \backslash \{i, j\}$, and $x \sim y$.

Strong invariance (SI): For all $x, y \in X$, $x \succsim y$ implies that for all $z \in \mathbb{R}^n$ and all $b \in \mathbb{R}_{++}$, we have $(bx + z) \succsim (by + z)$ whenever $(bx + z), (by + z) \in X$.

We claim that $\succsim$ is scalar continuous on X. Let $I = \{1, 2, 3, \ldots, n\}$. By using MIS, SI and WP, we can easily find a unique vector $(q_i)_{i \in I} \gg 0$ such that $e_i \sim q_i e$ for each $i \in I$. Thus by using SI, we can infer that for any $x \in X$, $x \sim \lambda(x)e$ where $\lambda(x) = \sum_{i \in I} q_i x_i$. Note that for all x in X we have $\lambda(x) \in [0, 1]$ since $\sum_{i \in I} q_i = 1$ and $x_i \in [0, 1]$ for every $i \in I$, and thus $\lambda(x) \in A(x) \cap B(x)$.

Since the preference order satisfies WP, we can infer that $\alpha(x) = \beta(x)$ for all x in X. To see this, let $\alpha(x) < \beta(x)$. Define $\varepsilon = \beta(x) - \alpha(x)$, and find $\delta \in (0, (\varepsilon/2))$, $\mu \in A(x)$ and $\eta \in B(x)$ such that $\mu < \alpha(x) + \delta$ and $\eta > \beta(x) - \delta$. Then, $\eta > \beta(x) - (\varepsilon/2) = [\beta(x) - \alpha(x)] + \alpha(x) - (\varepsilon/2) = \alpha(x) + (\varepsilon/2) > \mu$ so that we have, using the fact that $\succsim$ is monotone $x \succsim \eta e \succ \mu e \succsim x$ contradicting the transitivity of $\succsim$.

Since $\lambda(x) \in A(x) \cap B(x)$, we have a non-empty intersection of $A(x)$ and $B(x)$ and thus we must have $\lambda(x) = \alpha(x) = \beta(x)$ for each $x \in X$. Thus $\alpha(x) \in A(x)$ and $\beta(x) \in B(x)$ and so $A(x) = [\lambda(x), 1]$ and $B(x) = [0, \lambda(x)]$ by (2) and (3). This shows that $A(x)$ and $B(x)$ are closed in Y and therefore the preference order is scalar continuous, and has a representation by using Proposition 2.

Mitra and Ozbek (2010) show that whenever a preference order satisfies MIS, Invariance (a weaker form of SI), WP, and has a representation, then it also has a weighted utilitarian representation. Thus Proposition 2 together with the conditions above, ensures a weighted utilitarian representation for the preference ordering.

Example 3: Diamond's representation on infinite utility streams

The framework for analysis of social preference orders on infinite utility streams was introduced by Koopmans (1960). Diamond (1965) established the existence of a representation for monotone preference orders, which satisfy weak Pareto and sup-norm continuity. We now show how his existence result can be derived from Proposition 2.[13]

Let $X = Y^{\infty}$ and let $\succsim$ be a monotone preference order on X satisfying WP, and:

Sup-norm continuity: For each $x \in X$, the Lower and Upper Contour Sets, $LC(x) = \{y \in X : x \succsim y\}$ and $UC(x) = \{y \in X : y \succsim x\}$ respectively, are closed with respect to the sup-norm.

We claim that $\succsim$ satisfies scalar continuity. Let $x \in X$. Then the two sets $UC(x) \cap C$ and $LC(x) \cap C$ are closed in the sup-norm topology since C, the set of constant sequences in X, is also a closed set in the sup-norm topology. One can easily verify that the function $\pi : [0, 1] \to C$, defined as $\pi(k) = ke$ for every $k \in [0, 1]$, is continuous. Then we have $A(x) = \pi^{-1}(UC(x) \cap C)$ and $B(x) = \pi^{-1}(LC(x) \cap C)$ showing that $A(x)$ and $B(x)$ are closed sets in $[0, 1]$. Thus $\succsim$ satisfies scalar continuity and so $\succsim$ is representable by Proposition 2.

Example 4: Asheim–Mitra–Tungodden representation on infinite utility streams

Preference orders on infinite utility streams, which can be represented, exhibit a conflict if the order is required to satisfy certain equity and efficiency axioms simultaneously (See Diamond 1965, Basu and Mitra 2003, Hara et al. 2008, and others).

Seeking a way out of such impossibility results, Asheim et al. (2012) introduce weak versions of efficiency and equity, together with a weak continuity requirement to establish a class of sustainable recursive social welfare functions for monotone preference orders. In doing this, they first show the existence of representation for the preference orders by using a continuity condition called "restricted continuity" which is weaker than the usual sup–norm continuity. We show here how one can derive the existence of a representation by appealing to Proposition 2.

Let $X = Y^{\infty}$ and let $\succsim$ be a monotone preference order on X satisfying the following continuity condition:

Restricted continuity: For all $x, y \in X$, if x satisfies $x_t = z$ for all $t > 1$, and the sequence streams $\{x^n\}_{n \in \mathbb{N}}$ satisfies $\lim_{n \to \infty} \sup_t |x_t^n - x_t| = 0$ with, for each $n \in \mathbb{N}$, $x^n \succsim y$ (resp. $y \succsim x^n$), then $x \succsim y$ (resp. $y \succsim x$).

We claim that $\succsim$ satisfies scalar continuity. Let $x \in X$ and consider set $A(x)$. By definition of $\alpha(x)$, there exists a sequence $\{\lambda^s\}_{s \in \mathbb{N}}$ in $A(x)$ such that $\lambda^s \to \alpha(x)$ as $s \to \infty$. Define $x^s = \lambda^s e \in X$ for all $s \in \mathbb{N}$. Then by definition of set $A(x)$, we have $x^s \succsim x$ for all $s \in \mathbb{N}$. Moreover, as $s \to \infty$, $x^s \to \alpha(x)e$ in sup-norm metric. Thus by restricted continuity $\alpha(x)e \succsim x$ and so $\alpha(x) \in A(x)$ using definition of

[13] This approach coincides with Yaari's given in a footnote in Diamond (1965).

set $A(x)$. Since the order $\succsim$ is monotone, we must have $A(x) = [\alpha(x), 1]$ which is a closed set in [0, 1]. Following a similar argument for set $B(x)$, one can show that $B(x) = [0, \beta(x)]$ which is also a closed set in [0, 1]. Thus $\succsim$ satisfies scalar continuity and so by Proposition 2, $\succsim$ is a representable preference order.

6 Conclusion

In this paper, we have investigated the relation between scalar continuity and representability of monotone preference orders in a sequence space. Scalar continuity is shown to be sufficient for representability of a monotone preference order (Lemma 1). Generalizing this result, we have shown that a countable scalar discontinuity condition ensures representability of a monotone preference order (Theorem 1). Although these conditions are not necessary conditions for representability of a monotone preference order, they are very useful for applications. We have demonstrated this by indicating how some of the well-known representation results from the literature follow from our representation results established in Lemma 1 and Theorem 1. Moreover, we have related the countable scalar discontinuity condition to the well-known order dense property (Remark 1), which is both necessary and sufficient for representability.

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Appendix: Scalar continuity, restricted continuity and sup-norm continuity

In this section, we show that (i) there are monotone preference orders which satisfy scalar continuity, but violate restricted continuity and (therefore) sup-norm continuity (Example 1); and (ii) there are monotone preference orders which satisfy restricted continuity but violate sup-norm continuity (Example 2).

Example 1 We first construct the example without specifying any dimension for the sequence space, but we later indicate which case of it we are using, finite or infinite, when we are considering the relevant conditions for that case.

Let $(q_n)_{n \leq M}$ be a sequence for some $M \in \mathbb{N} \cup \{\infty\}$ satisfying:

$$q_n > 0 \quad \text{for all } n \leq M \text{ and } \sum_{n=1}^{M} q_n = 1 \tag{A1}$$

Let $f : X \to \mathbb{R}$ be defined by:

$$f(x) = \begin{cases} \sum_{n=1}^{M} q_n x_n & \text{if } x_n > 0 \text{ for all } n \leq M \\ 0 & \text{otherwise} \end{cases} \tag{A2}$$

Note that the sum in the definition of f in (A2) converges for any $x \in X$ and thus f is well-defined. Define $\succsim$ by:

$$\text{For all } x, y \in X,\ x \succsim y \text{ if and only if } f(x) \geq f(y) \tag{A3}$$

Then, $\succsim$ is a preference ordering on X, and f is a real-valued representation of it. Since f is increasing, [that is, for any $x, y \in X$, $x \geq y$ implies $f(x) \geq f(y)$ and $x \gg y$ implies $f(x) > f(y)$] by (A3) we have for any $x, y \in X$, if $x \geq y$, then $x \succsim y$ and if $x \gg y$, then $x \succ y$ and thus, $\succsim$ is monotone and it also satisfies WP.

Scalar continuity: Let $x \in X$. Then by (A2) and (A3), $A(x) = [f(x), 1]$ and $B(x) = [0, f(x)]$ which are both closed sets in [0, 1] showing that $\succsim$ satisfies scalar continuity.

Restricted continuity: We now show that $\succsim$ does not satisfy restricted continuity on $X = Y^\infty$. Let $z = (1 - q_1)e$ and consider the sequence $\{x^n\}_{n=1}^\infty$ in X where $x^n = (\frac{1}{n}, 1, 1, 1, \ldots)$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$, we have $f(x^n) = (1 - (\frac{n-1}{n})q_1) \geq (1 - q_1) = f(z)$ implying by (A3) that:

$$x^n \succsim z \text{ for all } n \in \mathbb{N} \tag{A4}$$

Let $x = (0, 1, 1, 1, \ldots)$ in X and note that:

$$d(x^n, x) = \frac{1}{n} \to 0 \text{ as } n \to \infty \tag{A5}$$

We have by (A2), $f(x) = 0 < (1 - q_1) = f(z)$, and hence by (A3):

$$z \succ x \tag{A6}$$

But then (A4), (A5) and (A6) imply that $\succsim$ on Y^∞ violates the restricted continuity condition.

Sup-norm continuity: Since continuity of an order in sup-norm topology implies restricted continuity, we infer from the result above that $\succsim$ on $X = Y^\infty$ does not satisfy sup-norm continuity.

We now observe that Example 1 also demonstrates that representation of the preference order in this example cannot be obtained by applying the representation results of Wold (1943), Diamond (1965), d'Aspremont and Gevers (2002), and Asheim et al. (2012).

Since in this example, restricted continuity and therefore sup-norm continuity, is violated, the representation results of Asheim et al. (2012) and Diamond (1965) are not applicable. We verify below that the example also violates Wold's continuity condition and Strong Invariance, so that the representation results of Wold (1943) and d'Aspremont and Gevers (2002) are also not applicable.

Wold: Let $\succsim$ be an order on $X = Y^n$ defined in (A3) for some given $n \in \mathbb{N}$. Let $x = e \in X$, $y = (1 - q_1)e \in X$ and $z = (e - e_1) \in X$. Then, by (A2), we have $f(x) = 1 > (1 - q_1) = f(y) > f(z) = 0$ and so by (A3), $x \succ y \succ z$. Let $a \in (0, 1)$ and define $w(a) = ax + (1 - a)z$. We have $w(a) = (e - (1 - a)e_1)$. By (A2), we get $f(w(a)) = (1 - (1 - a)q_1) > (1 - q_1) = f(y)$ and so by (A3), $w(a) \succ y$. This shows that there is no $b \in (0, 1)$ such that $y \succ w(b)$. Thus, the order $\succsim$ does not satisfy the condition of Wold.

Strong invariance: Let $\succsim$ be an order on $X = Y^n$ defined in (A3) for some given $n \in \mathbb{N}$. Let $x = (e - e_1) \in X$, $y = ((1 - q_1)/2)e \in X$, $b = 1$ and $z = (1/2)e_1 \in \mathbb{R}^n$. Then we have $(x+z) = (e-(1/2)e_1) \in X$ and $(y+z) = ((1-q_1)/2)e+(1/2)e_1 \in X$. Moreover, by (A2) $f(y) = ((1 - q_1)/2) > 0 = f(x)$ and so by (A3), $y \succ x$. Similarly, by (A2), $f(x + z) = 1 - (1/2)q_1 > ((1 - q_1)/2) + (1/2)q_1 = f(y + z)$ and so by (A3), $(x + z) \succ (y + z)$ showing that SI is not satisfied.

Example 2 Let $X = Y^\infty$ and consider a sequence $(q_n)_{n\in\mathbb{N}}$ defined as in (A1). Let $g : X \to \mathbb{R}$ be defined by:

$$g(x) = \begin{cases} \sum_{n=1}^{\infty} q_n x_n & \text{if } x_n > 0 \text{ for all } n > 1 \\ q_1 x_1 & \text{otherwise} \end{cases} \tag{A7}$$

Note that the sum in the definition of g in (A7) converges for any $x \in X$ and thus g is well-defined. Define $\succsim$ by:

$$\text{For all } x, y \in X, x \succsim y \quad \text{if and only if } g(x) \geq g(y) \tag{A8}$$

Then, $\succsim$ is a preference ordering on X, and g is a real-valued representation of it. Since g is increasing [that is, for any $x, y \in X$, $x \geq y$ implies $g(x) \geq g(y)$ and $x \gg y$ implies $g(x) > g(y)$], by (A8) we have for any $x, y \in X$, if $x \geq y$, then $x \succsim y$ and if $x \gg y$, then $x \succ y$ and thus, $\succsim$ is monotone and it also satisfies WP.

Sup-norm continuity: We first show that $\succsim$ does not satisfy sup-norm continuity on $X = Y^\infty$. Let $z = (1 - q_2)e$ and consider the sequence $\{x^n\}_{n=1}^{\infty}$ in X where $x^n = (1, \frac{1}{n}, 1, 1, \ldots)$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$, we have $g(x^n) = (1 - q_2) + (1/n)q_2 \geq 1 - q_2 = g(z)$ implying by (A8) that:

$$x^n \succsim z \text{ for all } n \in \mathbb{N} \tag{A9}$$

let $x = (1, 0, 1, 1, \ldots)$ in X and note that:

$$d(x^n, x) = \frac{1}{n} \to 0 \text{ as } n \to \infty \tag{A10}$$

We have by (A1) and (A7), $g(x) = q_1 < 1 - q_2 = g(z)$, and hence by (A8), we get:

$$z \succ x \tag{A11}$$

But then (A9), (A10) and (A11) imply that $\succsim$ on Y^∞ violates the sup-norm continuity condition.

Restricted continuity: We now show that the order $\succsim$ satisfies the restricted continuity condition. To see this, let $y \in X$ and $\{x^n\}_{n=1}^{\infty}$ be a sequence in X with $x^n \succsim y$ for all $n \in \mathbb{N}$ converging in sup-norm to some $x \in X$ such that for all $m > 1$, $x_m = a$ for some $a \in [0, 1]$ (A similar line of argument can be given for the case where $y \succsim x^n$ for all $n \in \mathbb{N}$). There are two cases to consider: either (i) there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $x_m^n > 0$ for all $m > 1$ or (ii) for every $N \in \mathbb{N}$ there exists $n \geq N$ such that $x_{m(n)}^n = 0$ for some $m(n) \in \mathbb{N}$ with $m(n) > 1$.

In case (i), by (A7) and (A8) we have $g(x^n) = qx^n \geq g(y)$ for all $n \geq N$. Since qx is sup-norm continuous, we have $qx^n \to qx$ as $n \to \infty$ and so $qx \geq g(y)$. Note that since for all $m > 1$, $x_m = a$ for some $a \in [0, 1]$, we have $g(x) = qx$ and so we must have $g(x) \geq g(y)$ implying that $x \succsim y$.

In case (ii), we can find a subsequence $\{x^{n_k}\}_{k=1}^{\infty}$ such that $x_{m_k}^{n_k} = 0$ for some $m_k \in \mathbb{N}$ with $m_k > 1$. Thus for all $k \in \mathbb{N}$, by (A7) and (A8) we have $g(x^{n_k}) = q_1 x_1^{n_k} \geq g(y)$. Since we have $x^{n_k} \to x$ as $k \to \infty$ in sup-norm, we must have $x_1^{n_k} \to x_1$ as $k \to \infty$ and thus $q_1 x_1^{n_k} \to q_1 x_1$ as $k \to \infty$. Therefore, by (A1), (A7) and (A8) we get $g(x) \geq q_1 x_1 \geq g(y)$ inferring that $x \succsim y$.

Scalar continuity: Note that since the monotone order $\succsim$ on $X = Y^{\infty}$ satisfies restricted continuity, it must also satisfy scalar continuity following the discussion given in Example 4 of Sect. 5.

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